ON THE COMMUTATIVITY OF THE BOUNDARY AND INTERIOR OPERATORS IN A TOPOLOGICAL SPACE.—N. Levine [2] discovered that, in a topological space, the interior and closure operators will commute if and only if the set on which they operate is the symmetric difference of a set that is both open and closed and a set that is nowhere dense. The intent of this paper is to characterize those sets for which the interior and boundary operators will commute.

The following notation will be used:

cA—closure of A
CA—complement of A
Int A—interior of A
BA—boundary of A
$A \nabla B$—symmetric difference of A and B

Lemma 1: $\{X,T\}$ is a topological space and A and E subsets of X, then $\mathrm{Int}(A \cap E) = \mathrm{Int}A \cap \mathrm{Int}E$.

Lemma 2: If $\{X,T\}$ is a topological space, A and E are subsets of X, and A is open and E is dense, then $c(A \cap E) = cA \cap cE = cA$. This is an exercise on p. 57 in Kelley's book (1955).

Lemma 3: If $\{X,T\}$ is a topological space and A is a subset of X, then $\mathrm{IntBA} \cap \mathrm{BIntA} = \emptyset$.

Proof: $\mathrm{IntBA} \cap \mathrm{BIntA} = \mathrm{Int}(cA \cap cCA) \cap c\mathrm{IntA} \cap cC\mathrm{IntA} = \mathrm{IntcA} \cap \mathrm{IntcCA} \cap c\mathrm{IntA} \cap cC\mathrm{IntA} = \mathrm{IntcA} \cap Cc\mathrm{IntA} \cap c\mathrm{IntA} \cap cC\mathrm{IntA} = \emptyset$.

Lemma 4: If $\{X,T\}$ is a topological space and A is a subset of X, then $\mathrm{IntBA} = \mathrm{BIntA}$ if and only if $\mathrm{IntcA} = c\mathrm{IntA} = \mathrm{IntA}$.

Proof: Necessity. Suppose $\mathrm{IntBA} = \mathrm{BIntA}$. Then these sets must both be empty in order to be equal since by lemma 2 they have nothing in common. Thus

(1) $\mathrm{IntBA} = \mathrm{IntcA} \cap Cc\mathrm{IntA} = \emptyset$ and

(2) $\mathrm{BIntA} = c\mathrm{IntA} \cap cCA = \emptyset$

Equations 1 and 2 imply $\mathrm{IntcA} \subset c\mathrm{IntA}$ and $c\mathrm{IntA} \subset CcCA = \mathrm{IntA}$ respectively. Since $\mathrm{IntA} \subset \mathrm{IntcA}$, it follows that $\mathrm{IntA} = \mathrm{IntcA} = c\mathrm{IntA}$.

Sufficiency. Suppose $\mathrm{IntcA} = c\mathrm{IntA} = \mathrm{IntA}$. This implies that $\mathrm{BIntA} = \emptyset$ and $\mathrm{IntBA} = \emptyset$, and thus the two sets are equal.

Theorem: If $\{X,T\}$ is a topological space and A is a subset of X, then $\mathrm{IntBA} = \mathrm{BIntA}$ if and only if $A = E \cup P$, where E is open and closed, P is nowhere dense, and $E \cap P = \emptyset$.

Proof: Using lemma 4, the proof reduces to showing that $\mathrm{IntcA} = c\mathrm{IntA} = \mathrm{IntA}$ if and only if $A = E \cup P$, where E is open and closed, P is nowhere dense, and $E \cap P = \emptyset$.

Necessity. Suppose $\mathrm{IntcA} = c\mathrm{IntA} = \mathrm{IntA}$. By Levine's theorem (Levine, 1961), it follows that if $\mathrm{IntcA} = c\mathrm{IntA}$, then $A = E \nabla P$ where E is open and closed and P is nowhere dense. Thus it is left to establish what further conditions the second equality places on E and P. In Levine's proof, $E = c\mathrm{IntA}$. Thus $E = \mathrm{IntA}$(i.e. $E = \mathrm{Int}(E \nabla P)$). $\mathrm{Int}(E \nabla P) = CcC[\, (E \cap CP) \cup (CE \cap P)\,] = Cc[\, (CE \cap CP) \cup (P \cap E)\,] = C[c(CE \cap CP) \cup c(P \cap E)\,]$. By lemma 2 and the fact that CP is dense it follows that $C[c(CE \cap CP) \cup c(P \cap E)\,] = CcCE \cap Cc(P \cap E) = E \cap Cc(P \cap E)$. Therefore, $E = E \cap Cc(P \cap E)$ which implies $P \cap E = \emptyset$.

Sufficiency. Suppose $A = E \cup P$, E is open and closed, P is nowhere dense, and $E \cap P = \emptyset$. By Levine's theorem, $c\mathrm{IntA} = \mathrm{IntcA}$. $\mathrm{IntA} = \mathrm{Int}(E \cup P) = Cc(CE \cap CP) = CcCE = E$. Thus IntA is closed and it follows that $C\mathrm{IntA} = \mathrm{IntcA} = \mathrm{IntA}$.—

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